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Dedicated to the memory of S.I. Pohozaev

On the Curve of Critical Exponents for Nonlinear Elliptic Problems in the Case of a Zero Mass

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Abstract—For semilinear elliptic equations $-\Delta u = \lambda|u|^{p-2}u-|u|^{q-2}u$, boundary value problems in bounded and unbounded domains are considered. In the plane of exponents $p \times q$, the so-called curves of critical exponents are defined that divide this plane into domains with qualitatively different properties of the boundary value problems and the corresponding parabolic equations. New solvability conditions for boundary value problems, conditions for the stability and instability of stationary solutions, and conditions for the existence of global solutions to parabolic equations are found.

Keywords: critical exponent, Pohozaev's identity, fibering method, stability of solutions.
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1. INTRODUCTION

We consider the equation

$$-\Delta u = \lambda|u|^{p-2}u - \mu|u|^{q-2}u \quad \text{in} \quad D. \tag{1.1}$$

Here $p > 0$, $q > 0$, $\lambda \geq 0$, $\mu \geq 0$, and D is one of the domains $D = \mathbb{R}^N$, $D = \Omega$, $D = \mathbb{R}^N \setminus \overline{\Omega}$, where Ω is a bounded star-shaped domain in $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$ and $N \geq 1$. We investigate the weak solutions $u \in W(D) := \mathcal{D}^{1,2}(D) \cap L^\infty(D)$ satisfying the following boundary conditions:

in the case $D = \Omega$,

$$u = 0 \quad \text{on} \quad \partial\Omega; \tag{1.2}$$

in the case $D = \mathbb{R}^N$,

$$|u(x)| \to 0 \quad \text{as} \quad |x| \to \infty; \tag{1.3}$$

and in the case $D = \mathbb{R}^N \setminus \overline{\Omega}$,

$$u = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad |u(x)| \to 0 \quad \text{as} \quad |x| \to \infty. \tag{1.4}$$

The boundary value problems (1.1), (1.2), (1.1)–(1.3), and (1.1)–(1.4) have a variational structure with the Euler—Lagrange functional

$$E_\lambda(u) = \frac{1}{2}\int_D |\nabla u|^2 dx - \lambda\frac{1}{p}\int_D |u|^p dx + \mu\frac{1}{q}\int_D |u|^q dx, \quad u \in W(D). \tag{1.5}$$

Following [3], we say that Eq. (1.1) corresponds to the zero mass case. Such an equation is the limiting case of the family of equations with nonzero masses, i.e., Eqs. (1.1) in which Δu is replaced by $(\Delta u - mu)$

with $m > 0$. In the case $\mu = 0$, Eq. (1.1) subject to one of the boundary conditions (1.2), (1.3), (1.4) provides an example of the classical boundary value problem that has a critical exponent. The study of nonlinear problems with critical exponents was started in the 1960s by Pohozaev and Fujita (see [2, 3]). In [2], the parabolic problem $u_t = \Delta u + \lambda |u|^{p-2}u$ in $D = \mathbb{R}^N$ was considered, where the exponent $p_F = \dfrac{2(N+1)}{N}$ is critical in the sense that, for $p \in (1, p_F)$, the parabolic problem has no nonnegative global solutions, while such solutions can exists when $p > p_F$. In [3], it was shown for the elliptic problem (1.1)–(1.3) for $\mu = 0$ and $N \geq 3$ that the existence of positive solutions is possible only for $p \in (2, 2^*]$, where 2^* is the Sobolev critical exponent ($2^* = \dfrac{2N}{N-2}$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$). Presently, the theory of critical exponents is a central topic in nonlinear analysis. The interest in this topic is caused both by the internal logic of the development of the theory of nonlinear differential equations and by the demands of applications (e.g., see [4–10] and references therein).

In general, the critical exponent can be defined as a value p^* that separates the intervals of the exponents p for which the equation has distinctive qualitative properties. The problem becomes more complicated if we consider Eq. (1.1) as a family of equations parameterized by two exponents $(p, q) \in \mathbb{R}^2$. In this case, we face a more general *problem on the existence of a curve of critical exponents*, i.e., a curve that separates the $p \times q$ domains on the plane in which Eq. (1.1) has qualitatively distinct properties. This problem was studied in [11], where a new curve of critical exponents $\mathscr{C}(N) \subset \mathscr{E}_0$ was found in $\mathscr{E}_0 := \{(p,q) \in \mathbb{R}^2 : 1 < q < p < 2\}$ that separates in $\mathscr{E}_0$ the domains in which the initial–boundary value parabolic problem associated with (1.1), (1.2) can have only stable or unstable ground states, depending on which domain contains the exponents (p, q). In this paper, we elaborate theses studies (see [11]) as applied to the entire quadrant $\mathscr{E} := \{(p,q) : p > 0, q > 0\}$.

As in [11], our approach essentially uses the function $\mathbb{R}^+ \ni r \mapsto E(ru)$ for $u \in W$, which is called the *fibering function* following the works by Pohozaev [12, 13]. Note that, in the case of equations with only one exponent, e.g., in the case $\mu = 0$ or $\lambda = 0$ in Eq. (1.1), the fibering function $E(ru)$, for all $u \in W \backslash 0$ has a unique stationary point $r_u > 0$ $(dE(r_u u)/dr = 0)$ of the same type for all $u \in W \backslash 0$ (i.e., either $d^2E(r_u u)/dr^2 < 0$ or $d^2E(r_u u)/dr^2 > 0$). If Eq. (1.1) is considered depending on two exponents (p, q), then $E(ru)$ can have two nonzero stationary points of different types or have no such points. Under the approach proposed in this paper, we overcome this difficulty as follows. To investigate the solvability of problems, in addition to the fibering equation $dE(ru)/dr = 0$ and Pohozaev's identity as is done in the conventional approach [3], we include in the analysis the equation $d^2E(ru)/dr^2 = 0$. An additional advantage of this approach is that it not only helps find necessary conditions for the existence of solutions but also makes it possible to determine the type of the stationary point of the fibering function $E(ru)$ they can correspond to $d^2E(ru)/dr^2 < 0$, $d^2E(ru)/dr^2 > 0$, or $d^2E(ru)/dr^2 = 0$. The ability to classify these properties of solutions allow us to investigate the stability of these solutions for the corresponding nonstationary problems (see Lemmas 4.1, 5.2, 5.5, and 5.8).

The central role in the present paper is played by the mapping

$$d^*(p,q) = N(p-2)(q-2) - 2pq, \quad (p,q) \in \mathbb{R}^2. \tag{1.6}$$

The set $\mathscr{C}(p,q) := \{(p,q) \in \mathbb{R}^2 : d^*(p,q) = 0\}$ is called the *curve of critical exponents*. This curve and the curves $p = 2$, $q = 2$, $p = 2^*$, $q = 2^*$, and $p = q$ divide the plane of the exponents into domains as shown in Figs. 1–3. The main aim of this paper is to investigate the properties of problems (1.1), (1.2), (1.3), (1.4) depending on the domain containing the exponents (p, q). In Section 2, we derive necessary conditions for the existence of solutions to problems (1.1), (1.2), (1.3), (1.4) and give a classification of solutions depending on the type of the stationary points of the fibering function corresponding to them (see Theorems 2.2, 2.4, and 2.5). As a consequence of these results, we give an answer (in the case of equations with a zero mass) to the Strauss problem (see [14]) of the solvability of problem (1.1)–(1.3) for $2^* < q < p$ and $N \geq 3$. In Section 3, we investigate the existence of solutions. The main result is Theorem 3.6. In Section 4, we use qualitative results on the types of stationary points of the fibering function to obtain results on the lin-

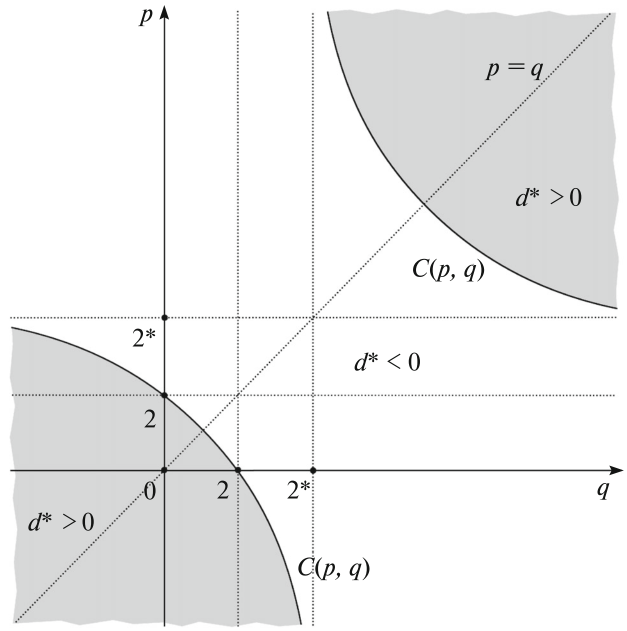

Fig. 1. The curve of critical exponents for $N \geq 3$.

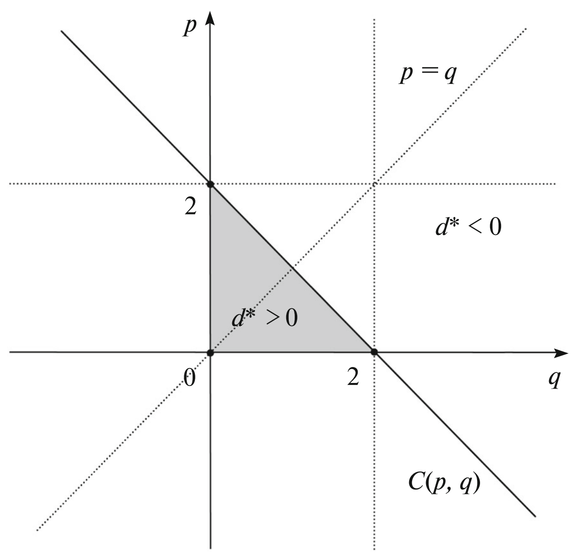

Fig. 2. The curve of critical exponents for $N = 2$.

ear instability of stationary solutions to parabolic equations. In Section 5, we investigate the stability of solutions to parabolic equations. Here, we elaborate the results obtained in [11] for the case of exponents in the quadrant $\mathscr{E}$. Note that, by Derrick's theorem [15, 16], in the case $D = \mathbb{R}^N$, the solutions to problem (1.1)–(1.3) are linearly unstable stationary states for the corresponding parabolic equations. However, this result is valid only for $p > 2$, $q > 2$, while the case $p, q \in (1, 2]$ is not covered by Derrick's theorem. We show that, in the case $p, q \in (1, 2]$, a result that is generally opposite to the assertion of Derrick's theorem holds. In Lemma 5.8, we use the curve of critical exponents $\mathscr{C}(p, q)$ to find a subset $\mathscr{E}_u \subset (1, 2] \times (1, 2]$ possessing the following properties in the case $(p, q) \in \mathscr{E}_u$: the parabolic problem associated with (1.1)–(1.3), has global solutions and stable (in the sense specified) below stationary states.

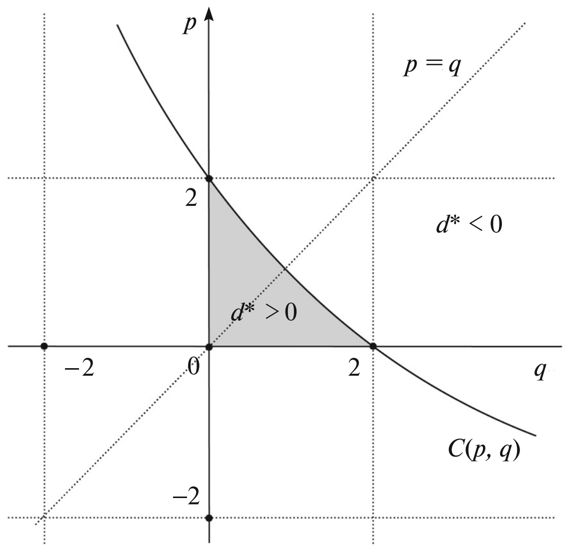

Fig. 3. The curve of critical exponents for $N = 1$.

2. THE CURVE OF CRITICAL EXPONENTS

It is sufficient to examine Eq. (1.1) depending only on one of the parameters $\lambda > 0$ or $\mu > 0$. Indeed, making the change of variables $\tilde{u} = tu$ in Eq. (1.1), e.g., with $t = (1/\mu)^{1/(q-2)}$, we obtain Eq. (1.1) with $\tilde{\mu} = 1$ and $\tilde{\lambda} = \lambda(1/\mu)^{(p-2)/(q-2)}$. Furthermore, in the case $D = \mathbb{R}^N$, problem (1.1)–(1.3) is actually independent of both parameters μ and λ. Indeed, if u satisfies (1.1)–(1.3) at a certain $\lambda > 0$, then we can make the change of variables $v_{\tau,\sigma} = \tau u(\sigma x)$ with $\tau = \lambda^{1/(p-q)}$ and $\sigma^2 = \lambda^{(q-2)/(p-q)}$ to obtain a solution to Eq. (1.1) with $\tilde{\lambda} = 1$. For this reason, we assume that $\mu = 1$ below. In addition, while considering Eq. (1.1) in the case $D = \mathbb{R}^N$, we assume (unless otherwise indicated) that $\lambda = 1$ and omit the coefficient λ.

Let $D = \mathbb{R}^N$, $D = \Omega$, or $D = \mathbb{R}^N\setminus\overline{\Omega}$, where Ω is a bounded domain in $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. We define $W(D) := \mathcal{D}^{1,2}(D) \cap L^\infty(D)$, where $\mathcal{D}^{1,2}(D)$ is the Hilbert space defined as the completion of $C_0^\infty(D)$ with respect to the norm $\|w\|_1 = \left(\int_D |\nabla w|^2 dx\right)^{1/2}$. In this notation, the boundary value problems (1.1), (1.2), (1.3), and (1.4) are written in the unified form

$$-\Delta u = \lambda|u|^{p-2}u - |u|^{q-2}u, \quad u \in W(D), \tag{2.1}$$

where the equality is interpreted in the weak sense. Let

$$T(u) := \int_D |\nabla u|^2 dx, \quad A(u) = \int_D |u|^p dx, \quad B(u) = \int_D |u|^q dx, \quad u \in W(D).$$

Then, $E_\lambda(u) = \frac{1}{2}T(u) - \lambda\frac{1}{p}A(u) + \frac{1}{q}B(u)$. A weak solution $u \neq 0$ to problem (2.1) is called the *ground state* [1] if $E_\lambda(u) \le E_\lambda(w)$ for any other weak solution $w \in W(D)\setminus 0$ to this problem. Consider the fibering function

$$E_\lambda(ru) = \frac{r^2}{2}T(u) - \lambda\frac{r^p}{p}A(u) + \frac{r^q}{q}B(u), \quad r \in \mathbb{R}^+, \quad r > 0, \quad u \in W(D)\setminus 0.$$

Introduce the notation $E'_\lambda(u) := \frac{d}{dr}E_\lambda(ru)\Big|_{r=1}$, $E''_\lambda(u) := \frac{d^2}{dr^2}E_\lambda(ru)\Big|_{r=1}$. Note that, if u is a solution to problem (2.1), then $E'_\lambda(u) = 0$; i.e., $r = 1$ is a stationary point of the function $E_\lambda(ru)$. It is easily seen that, for each $u \in W(D)\backslash 0$, the following conditions are fulfilled.

Condition 1. If $0 < p < \min\{2,q\}$ or $\max\{2,q\} < p$, then $E_\lambda(ru)$ $\forall\lambda \geq 0$ has a unique stationary point $r > 0$; furthermore, $E''_\lambda(ru) > 0$ if $1 < p < \min\{2,q\}$, and $E''_\lambda(ru) < 0$ if $\max\{2,q\} < p$.

Condition 2. If $1 < q < p < 2$ or $2 < p < q$, then there exists a $\lambda_u > 0$ such that, $\forall\lambda \in (0,\lambda_u)$, $E_\lambda(ru)$ has no stationary points $r > 0$; for $\lambda = \lambda_u$, there exists a unique nonzero stationary point $r > 0$, and $E''_\lambda(ru) = 0$; for $\lambda \in (\lambda_u,+\infty)$, there exist two nonzero stationary points $r_{\max}$ and $r_{\min}$ such that $E''_\lambda(r_{\max}u) < 0$ and $E''_\lambda(r_{\min}u) > 0$.

Let $u \in W(D)$ be a weak solution to Eq. (2.1). Then, according to the standard regularity theory of solutions to elliptic equations (see [17]), we have $u \in C^2(D) \cap C^{1,\kappa}(\overline{D})$ for $\kappa \in (0,1)$. Hence, u satisfies Pohozaev's identity (see [3, 18])

$$P_\lambda(u) + a\frac{1}{2N}\int_{\partial D}\left|\frac{\partial u}{\partial \nu}\right|^2 x\cdot\nu\, ds = 0, \tag{2.2}$$

where $\nu := \nu(x)$ is the normal vector to the boundary at the point $x \in D$, $a = 0$ if $D = \mathbb{R}^N$, $a = 1$ if $D = \Omega$ or $D = \mathbb{R}^N\backslash\overline{\Omega}$, and

$$P_\lambda(u) := \frac{1}{2^*}T(u) - \lambda\frac{1}{p}A(u) + \frac{1}{q}B(u), \quad u \in W(D), \tag{2.3}$$

is Pohozaev's function. Here, $\frac{1}{2^*} = \frac{N-2}{2N}$ if $N \geq 3$, $\frac{1}{2^*} = 0$ if $N = 2$, and $\frac{1}{2^*} = -\frac{1}{2}$ if $N = 1$. In addition we will use the following notation:

$$\frac{(q-2^*)}{2^*q} := -\frac{1}{q}, \quad \frac{(p-2^*)}{2^*p} := -\frac{1}{p} \quad \text{if} \quad N = 2$$

and

$$\frac{(q-2^*)}{2^*q} := -\frac{q+2}{q}, \quad \frac{(p-2^*)}{2^*p} := -\frac{p+2}{p} \quad \text{if} \quad N = 1.$$

Proposition 2.1. *Let $u \in W(D)$ be a weak solution to Eq.* (2.1), *Ω be a bounded domain in $\mathbb{R}^N$ that is a star-shaped domain with respect to the origin of $\mathbb{R}^N$, and let $\partial\Omega$ be a C^1-smooth boundary. Then,*

(i) $P_\lambda(u) = 0$ *if* $D = \mathbb{R}^N$;

(ii) $P_\lambda(u) \leq 0$ *if* $D = \Omega$;

(iii) $P_\lambda(u) \geq 0$ *if* $D = \mathbb{R}^N\backslash\overline{\Omega}$.

Proof. Assertion (i) is an immediate consequence of formula (2.2). Note that, if Ω is a star-shaped domain with respect to the origin of $\mathbb{R}^N$, then $x\cdot\nu \geq 0$ for all $x \in \partial\Omega$. Hence, (2.2) implies (ii) and (iii).

Let E'_λ, E''_λ, $P_\lambda \in \mathbb{R}$. Consider the system of equations

$$\begin{gathered} T(u) - \lambda A(u) + B(u) = E'_\lambda, \\ \frac{1}{2^*}T(u) - \lambda\frac{1}{p}A(u) + \frac{1}{q}B(u) = P_\lambda, \\ T(u) - (p-1)\lambda A(u) + (q-1)B(u) = E''_\lambda, \end{gathered} \tag{2.4}$$

where $T(u)$, $\lambda A(u)$, $B(u) \in \mathbb{R}$ are considered as unknowns. The determinant of this system is

$$d = \frac{(q-p)(N(p-2)(q-2)-2pq)}{2Npq} = \frac{(q-p)}{2Npq}d^*(p,q), \tag{2.5}$$

where d^* is given by formula (1.6). In the case $d \neq 0$ and $E'_\lambda = 0$, the solution to system (2.4) has the form

$$\begin{aligned} T(u) &= \frac{1}{d}\frac{(q-p)}{pq}E''_\lambda + \frac{(q-p)}{d}P, \\ \lambda A(u) &= \frac{1}{d}\frac{(q-2^*)}{2^*q}E''_\lambda + \frac{(q-2)}{d}P, \\ B(u) &= \frac{1}{d}\frac{(p-2^*)}{2^*p}E''_\lambda + \frac{(p-2)}{d}P. \end{aligned} \tag{2.6}$$

In the case $D = \mathbb{R}^N$, we have the following result.

Theorem 2.2. *Let $D = \mathbb{R}^N$, $p \neq q$, $p > 0$, and $q > 0$. Then, the following is true.*

(1) *For the existence of a nonzero solution to problem* (2.1), *it is necessary that* $2^* < p < q$ *or* $0 < q < p < 2^*$ *if* $N \geq 3$, *and* $0 < q < p$ *if* $N = 1, 2$.

(2) *If $u \in W(D)$ is a weak solution to problem* (2.1), *then $E(u) > 0$ and $E''(u) = 0 \Leftrightarrow d^*(p,q) = 0$, $E''(u) > 0 \Leftrightarrow d^*(p,q) > 0$, and $E''(u) < 0 \Leftrightarrow d^*(p,q) < 0$.*

Proof. Suppose that there exists a weak nonzero solution u to problem (2.1). Then, $E'(u) = 0$ and, by Proposition 2.1, $P(u) = 0$. Therefore, (2.6) implies

$$\begin{aligned} d \cdot T(u) &= \frac{(q-p)}{pq}E''(u), \\ d \cdot A(u) &= \frac{(q-2^*)}{2^*q}E''(u), \\ d \cdot B(u) &= \frac{(p-2^*)}{2^*p}E''(u). \end{aligned} \tag{2.7}$$

Since $T(u)$, $A(u)$, $B(u) > 0$, these equalities are possible only if the factors $(q-p)$, $(q-2^*)$, and $(p-2^*)$ multiplying $E''(u)$ have identical signs. Hence, in the case $N \geq 3$, it must be $q > p$, $q > 2^*$, $p > 2^*$ or $q < p$, $q > 2^*$, $p < 2^*$; in the case $N = 1, 2$, it must be $q < p$. Therefore, assertion (1) of Theorem 2.2 is valid.

Note that, if $2^* < p < q$ and $N \geq 3$, then $(q-p) > 0$, $(q-2^*) > 0$, and $(p-2^*) > 0$. This and (2.7) imply that the sign of $E''(u)$ coincides with the sign of $d^*(p,q)$. It is easy to see that the same holds for $0 < q < p < 2^*$ and $N \geq 3$ or for $0 < q < p$ and $N = 1, 2$. Thus, we have assertion (2) of Theorem 2.2.

In [14], the question about the existence of solution to problem (1.1)–(1.3) in the case $2^* < q < p$ and $N \geq 3$ was posed. In the case of a zero mass, assertion (1) of Theorem 2.2 gives the following answer to this question.

Corollary 2.3. In the case $2^* < q < p$ and $N \geq 3$, problem (1.1)–(1.3) has no solutions.

In the case of a bounded domain $D = \Omega$, we have the following result.

Theorem 2.4. *Let $p \neq q$, $p > 0$, $q > 0$, and $D = \Omega$, where Ω is a bounded star-shaped domain with respect to the origin of $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, the following is true:*

(1) *for the existence of a nonzero solution to problem* (2.1), *it is necessary that, in the case* $N \geq 3$, $0 < p < q$ *or* $0 < q < p < 2^*$;

(2) *if $u \in W(D)$ is a weak nonzero solution to problem* (2.1), *then, in the case $N \geq 1, E''(u) > 0$ if $d^*(p,q) > 0$ or $0 < p < \min\{2,q\}$, and $E''(u) < 0$ if $\max\{2,q\} < p < 2^*$.*

Proof. Let u be a weak solution to problem (2.1), and let $N \geq 3$. Then, since $E'_\lambda(u) = 0$ and $P_\lambda(u) \leq 0$ by Proposition 2.1, (2.6) implies

$$d^* T(u) \leq \frac{1}{pq}E''(u), \tag{2.8}$$

$$\lambda\frac{d^*(q-p)}{(q-2)}A(u) \leq \frac{(q-2^*)}{2^*q(q-2)}E''(u), \tag{2.9}$$

$$\frac{d^*(q-p)}{(p-2)}B(u) \le \frac{(p-2^*)}{2^*p(p-2)}E''(u). \qquad (2.10)$$

Inequality (2.8) implies that, if $d^* > 0$, then $E''(u) > 0$. In this case, inequalities (2.9) and (2.10) are consistent because $d^* > 0$ implies $\max\{p,q\} < 2$ or $2^* < \min\{p,q\}$.

Consider the case $\max\{2,q\} < p$. Due to Condition 1, $E''(u) < 0$. Hence, (2.8) implies $d^* < 0$. Then, the inequality $d^*(q-p) > 0$ and (2.10) imply $p < 2^*$. Thus, for the existence of a solution in this case, it is necessary that $\max\{2,q\} < p < 2^*$. It is easy to verify that, in the case $0 < q < p < 2$, inequalities (2.8)–(2.10) are consistent. Summarizing the reasoning above, we conclude that, in the case $N \ge 3$, in the semi-plane $0 < q < p$, the condition $0 < q < p < 2^*$ is necessary for the existence of a solution to problem (2.1).

Consider the case $N = 2$. Then, (2.6) and the inequality $P_\lambda(u) \le 0$ imply that

$$d^*T(u) \le \frac{1}{pq}E''(u), \quad \lambda\frac{d^*(q-p)}{(q-2)}A(u) \le -\frac{1}{q(q-2)}E''(u), \quad \frac{d^*(q-p)}{(p-2)}B(u) \le -\frac{1}{p(p-2)}E''(u). \qquad (2.11)$$

The first inequality in (2.11) implies that $E''(u) > 0$ if $d^* > 0$. Furthermore, $d^* > 0$ implies $p < 2$ and $q < 2$. Using this fact, we conclude that these inequalities are consistent for all $0 < p$ and $0 < q$ under consideration. In the case $N = 1$, the reasoning is similar.

In the case $D = \mathbb{R}^N \backslash \Omega$, the following result holds.

Theorem 2.5. *Let $p \ne q$, $p > 0$, $q > 0$, and $D = \mathbb{R}^N \backslash \Omega$, where Ω is a bounded star-shaped domain with respect to the origin of $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, the following is true*:

(1) *for the existence of a nonzero solution to problem* (2.1), *it is necessary that, in the case $N \ge 3$, $0 < q < p$ or $2^* < p < q$ and, in the case $N = 1, 2$, $0 < q < p$ or $2 < p < q$*;

(2) *if $u \in W(D)$ is a weak nonzero solution to problem* (2.1), *then,*

in the case $N \ge 3$, $E''(u) < 0$ if $d^*(p,q) < 0$ and $2^* < p < q$ or $d^*(p,q) < 0$ and $0 < q < p$;

in the case $N = 1, 2$, $E''(u) < 0$ if $2 < p < q$ or $d^*(p,q) < 0$ and $0 < q < p$.

Proof. Let u be a weak solution to problem (2.1). Then, in the case $N \ge 3$, (2.6) with the inequality $P_\lambda(u) \ge 0$ taken into account implies

$$d^*T(u) \ge \frac{1}{pq}E''(u), \qquad (2.12)$$

$$\lambda\frac{d^*(q-p)}{(q-2)}A(u) \ge \frac{(q-2^*)}{2^*q(q-2)}E''(u), \qquad (2.13)$$

$$\frac{d^*(q-p)}{(p-2)}B(u) \ge \frac{(p-2^*)}{2^*p(p-2)}E''(u). \qquad (2.14)$$

Using (2.12), we conclude that, if $d^* \le 0$, then $E''(u) < 0$. Consider the case $0 < p < q$. Since $E''(u) > 0$ for $0 < p < \min\{2,q\}$, we conclude that, for $d^* \le 0$, problem (2.1) cannot have solutions. If $2 \le p < q$, then inequalities (2.13) and (2.14) can hold only for $2^* < p < q$. It is easy to verify that, in the case $2 < p < q$ and $d^* > 0$, inequalities (2.12)–(2.14) are consistent; and, in the case $0 < p < q < 2$ and $d^* > 0$, they are inconsistent. Therefore, if $0 < p < q$, then it is necessary that $2^* < p < q$, and if additionally $d^* \le 0$, then $E''(u) < 0$. The analysis of inequalities (2.12)–(2.14) for $0 < q < p$ shows that they are consistent.

The cases $N = 1$ and $N = 2$ are analyzed analogously to each other. By way of example, consider the case $N = 2$. Using (2.6) and taking into account $P_\lambda(u) > 0$, we have

$$d^*T(u) > \frac{1}{pq}E''(u), \qquad (2.15)$$

$$\lambda\frac{d^*(q-p)}{(q-2)}A(u) > -\frac{1}{q(q-2)}E''(u), \qquad (2.16)$$

$$\frac{d^*(q-p)}{(p-2)}B(u) > -\frac{1}{p(p-2)}E''(u). \qquad (2.17)$$

The analysis of these inequalities in the case $0 < q < p$ shows that they are consistent. It follows from (2.15) that $d^* \le 0$ implies $E''(u) < 0$. Note that, if $2 < p < q$, then $d^* \le 0$ and, therefore, the left-hand sides of inequalities (2.16) and (2.17) are negative, while their right-hand sides are positive. This contradiction proves that the inequalities are inconsistent. They are also inconsistent in the case $d^* \le 0$ and $0 < p < \min\{2, q\}$.

3. EXISTENCE OF SOLUTIONS

The existence of a solution to Eq. (2.1) in the case $D = \mathbb{R}^N$ follows from the results obtained in [1, 14, 19–22]. We summarize these results in the following lemma.

Lemma 3.1. *Let $D = \mathbb{R}^N$. Then, the following is true:*

(1) *if $N = 1, 2$ and $1 < q < p$, then there exists a solution $u \in C^2(\mathbb{R}^N)$ to problem* (2.1) *such that $u \ge 0$, $u(x) = u(r)$, $|x| = r$, $x \in \mathbb{R}^2$, and $u'(r) < 0$ for $r > 0$. In addition, if $2 < p < q$, then $u > 0$ in $\mathbb{R}^N$;*

(2) *if $N \ge 3$ and $2^* < p < q$ or $1 < q < p < 2^*$, then there exists a nonnegative solution $u \in W(D) \cap C^2(\mathbb{R}^N)$ to problem* (2.1) *that is a ground state. Furthermore, if $2^* < p < q$ or $2 < q < p < 2^*$, then $u > 0$ and the function u is spherically symmetric and monotonically decreasing, i.e., $u(x) = u(r)$, $|x| = r$ for $x \in \mathbb{R}^N$ and $u'(r) < 0$ for $r > 0$.*

The proof of assertion (1) in the case $2 < q < p$ follows from Theorem 1.1 in [22] in the case $N = 2$ and from Theorem 5 in [3] in the case $N = 1$. A proof of this assertion in the case $1 < q < p \le 2$ can be found in [19–21, 23].

The proof of existence in assertion (2) follows from the following theorem in [1] about the sufficient conditions for the existence of solutions.

Theorem 3.2 (Berestycki–Lions). *Let $N \ge 3$ and g: $\mathbb{R} \to \mathbb{R}$ be a continuous odd function satisfying the conditions*

(1) $\overline{\lim}_{s\to+0} \dfrac{g(s)}{s^l} \le 0$, *where $l = 2^* - 1$;*

(2) $\exists \zeta > 0$ *such that* $G(\zeta) := \int_0^{\zeta} g(s)ds > 0$;

(3) *if $g(s) > 0$ for all $s > \zeta_0 := \inf\{\zeta > 0 :\ G(\zeta) > 0\}$, then*

$$\lim_{s\to+\infty} \frac{g(s)}{s^l} = 0.$$

Then, there exists a nonnegative solution $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ to the problem

$$-\Delta u = g(u), \tag{3.1}$$

$$u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{3.2}$$

and $u \in C^2(\mathbb{R}^N)$.

It is easy to verify that the function $g(s) = |s|^{p-2}s - |s|^{q-2}s$ is continuous for $1 < \min\{p, q\}$. In addition, it satisfies conditions (1)–(3) of Theorem 3.2 if and only if $2^* < p < q$ or $1 < q < p < 2^*$. Note that this, in particular, implies the following result.

Corollary 3.3. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous odd function. Then, conditions (1)–(3) of Theorem 3.2 are necessary and sufficient for the existence of solutions to problem (3.1)–(3.2).

The second part of assertion (2) of Lemma 3.1 follows from the maximum principle for elliptic equations (e.g., see [1]).

Consider the case $D = \Omega$. In this case, we will consider only the exponents in the set $\max\{p, q\} < 2^*$. Then, by Sobolev's theorem, we have the embedding $\mathcal{D}^{1,2}(\Omega) \subset L^{\gamma}(D)$ for $\gamma \in (1, 2^*)$. Therefore, we may consider $\mathcal{D}^{1,2}(\Omega)$ as $W(D)$. We will construct a solution to (2.1) by the Nehari manifold method using a variational problem with an imposed constraint:

$$E_\lambda(u) \to \min, \quad u \in N_\lambda. \tag{3.3}$$

Here $\mathcal{N}_\lambda := \{u \in W(D)\backslash 0 : \; E'_\lambda(u) = 0\}$ is called the Nehari manifold. Let

$$\hat{E}_\lambda := \min\{E_\lambda(u) : \; u \in N_\lambda\}. \tag{3.4}$$

Note that, since every weak solution to problem (2.1) belongs to the Nehari manifold $\mathcal{N}_\lambda$, the minimizer $u_\lambda \in \mathcal{N}_\lambda$ of problem (3.3) satisfying Eq. (2.1) is a ground state.

Proposition 3.4. *Let* $\lambda > 0$ *and* u_λ *be a minimizer of problem* (3.3), *satisfying the condition*

$$E''_\lambda(u_\lambda) \neq 0. \tag{3.5}$$

Then, u_λ *is a weak solution to problem* (2.1), *i.e.,* $D_u E_\lambda(u_\lambda) = 0$.

Proof. A proof can be found, e.g., in [12, 13].

Let us find the values of λ at which $\mathcal{N}_\lambda \neq \emptyset$ and condition (3.5) is satisfied. To this end, we use the nonlinear Rayleigh quotient method [24]. Let $u \in \mathcal{D}^{1,2}(\Omega)\backslash 0$, $1 < \min\{p, q\}$, and $\max\{p, q\} < 2^*$; then, the following Rayleigh quotient is well defined:

$$R(u) = \frac{\int_D |\nabla u|^2 dx + \int_D |u|^q dx}{\int_D |u|^p dx}.$$

Consider

$$R(ru) = \frac{r^{2-p}\int_D |\nabla u|^2 dx + r^{q-p}\int_D |u|^q dx}{\int_D |u|^p dx}, \quad u \in D^{1,2}(\Omega)\backslash 0, \quad r > 0. \tag{3.6}$$

By differentiating this function with respect to r, we obtain

$$\frac{\partial}{\partial r} R(ru) = \frac{(2-p) r^{1-p}\int_D |\nabla u|^2 dx + (q-p) r^{q-p-1}\int_D |u|^q dx}{\int_D |u|^p dx}. \tag{3.7}$$

It is easy to verify that the equation $\frac{\partial}{\partial r} R(ru) = 0$ has a solution only if $1 < q < p < 2$ and $2 < p < q$; moreover, this solution is unique and is given by the formula

$$r_{\min}(u) = \left(\frac{(p-2)\int_D |\nabla u|^2 dx}{(q-p)\int_D |u|^q dx}\right)^{1/q-2}. \tag{3.8}$$

Substitute $r_{\min}(u)$ into $R(ru)$ to obtain the nonlinearly generalized Rayleigh quotient

$$\lambda(u) := R(r_{\min}(u)u) = c_{p,q}\frac{\left(\int_D |\nabla u|^2 dx\right)^{\frac{q-p}{q-2}}\left(\int_D |u|^q dx\right)^{\frac{p-2}{q-2}}}{\int_D |u|^p dx}, \tag{3.9}$$

where

$$c_{p,q} = \frac{(p+q-4)}{(q-2)}\left(\frac{(q-2)}{(p-2)}\right)^{\frac{p-2}{q-2}}.$$

Hence, we obtain the following critical value (see [24, 25]):

$$\lambda_{(p,q)} = c_{p,q} \inf_{u \in D^{1,2}(D)\setminus 0} \frac{\left(\int_D |\nabla u|^2 dx\right)^{\frac{q-p}{q-2}} \left(\int_D |u|^q dx\right)^{\frac{p-2}{q-2}}}{\int_D |u|^p dx}. \tag{3.10}$$

Using the same reasoning for the Rayleigh quotient

$$R_E(u) = \frac{\frac{1}{2}\int_D |\nabla u|^2 dx + \frac{1}{q}\int_D |u|^q dx}{\frac{1}{p}\int_D |u|^p dx},$$

we obtain

$$\lambda_E(u) := R_E(r_{\min}(u)u) = c'_{p,q} c_{p,q} \frac{\left(\int_D |\nabla u|^2 dx\right)^{\frac{q-p}{q-2}} \left(\int_D |u|^q dx\right)^{\frac{p-2}{q-2}}}{\int_D |u|^p dx}, \tag{3.11}$$

where $c'_{p,q} = (p/2)(2/q)^{\frac{p-2}{q-2}}$. Consider $\lambda_{E,(p,q)} = \inf_{u \in D^{1,2}(D)\setminus 0} \lambda_E(u)$. It is clear that $\lambda_{E,(p,q)} = c'_{p,q}\lambda_{(p,q)}$, and the following result holds.

Lemma 3.5. *Let $1 < q < p < 2$ or $2 < p < q < 2^*$, and $D = \Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, $0 < \lambda_{(p,q)} < +\infty$ and $\lambda_{(p,q)} < \lambda_{E,(p,q)}$. Moreover, $\mathcal{N}_\lambda \neq \emptyset$ if and only if $\lambda > \lambda_{(p,q)}$ and, if $\lambda > \lambda_{E,(p,q)}$, then there exists a $u \in \mathcal{D}^{1,2}(D)\setminus 0$ such that $E_\lambda(u) < 0$.*

Proof. The inequality $0 < \lambda_{(p,q)}$ is easily derived from the Hölder and Sobolev inequalities. The inequality $\lambda_{(p,q)} < \lambda_{E,(p,q)}$ follows from the fact that $c'_{p,q} > 1$. To prove the remaining part of the lemma, it suffices to note that $\mathcal{N}_\lambda = \{u \in W(D)\setminus 0 : \ R(u) = \lambda\}$ $\forall\lambda > 0$ and, if $R_E(u) < \lambda$ for a certain $u \in \mathcal{D}^{1,2}(\Omega)\setminus 0$, then $E_\lambda(u) < 0$.

Now we prove the following main result.

Theorem 3.6. *Let $N \geq 1$ and $D = \Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, the following holds*:

(1) *If $1 < p < \min\{2, q\}$ or $1 < q$ and $\max\{2, q\} < p < 2^*$, then, for all $\lambda \in (0, +\infty)$, there exists a solution $u \in W(D)$ to problem* (2.1) *such that u is a ground state, $u \in C^2(D) \cap C^{1,\kappa}(\overline{D})$ for a certain $\kappa \in (0,1)$, and $u_\lambda > 0$ in D.*

(2) *If $1 < q < p \leq 2$ or $2 < p < q < 2^*$, then problem* (2.1) *has no solutions for $0 < \lambda < \lambda_{(p,q)}$. If $\lambda > \lambda_{E,(p,q)}$, then there exists a solution $u_\lambda \in W(D)$ to problem* (2.1) *such that u_λ is a ground state, $u_\lambda \in C^2(D) \cap C^{1,\kappa}(\overline{D})$ for a certain $\kappa \in (0,1)$, $u_\lambda \geq 0$ in D, and $E_\lambda(u_\lambda) < 0$, $E''_\lambda(u_\lambda) > 0$ $\forall\lambda > \lambda_{E,(p,q)}$. In addition, if $2 < p < q < 2^*$, then $u_\lambda > 0$ in D.*

Proof. Note that, since $E_\lambda(u) = E_\lambda(|u|)$ and $|u| \in \mathcal{N}_\lambda$ $\forall u \in \mathcal{N}_\lambda$, the existence of a minimizer u_λ in problem (3.3) implies that $|u_\lambda|$ is a minimizer as well. Condition (3.5) is evidently retained in this case. Thus, by proving the existence of a minimizer u_λ in problem (3.3) that satisfies condition (3.5), we obtain the existence of a weak nonnegative solution to (2.1). Furthermore, if u_λ is a weak solution to problem (3.3), then the regularity theory of solutions to elliptic equations (see [17]) implies that $u_\lambda \in C^2(D) \cap C^{1,\kappa}(\overline{D})$ for a certain $\kappa \in (0,1)$. In addition, for $1 < p < \min\{2,q\}$, $1 < \max\{2,q\} < p < 2^*$, or $2 < p < q < 2^*$, the maximum principle for elliptic boundary value problems implies (see [26, 27]) that $u_\lambda > 0$ in D. Therefore, to prove assertions (1) and (2) of Theorem 3.2, it suffices to find (for the corresponding λ) a minimizing sequence in problem (3.3) satisfying condition (3.5).

Proof of assertion (1) of Theorem 3.6. Note that, if $1 < p < \min\{2,q\}$, then $E''_\lambda(u) > 0$ $\forall u \in \mathcal{N}_\lambda$ and, if $1 < q$ and $\max\{2,q\} < p < 2^*$, then $E''_\lambda(u) < 0$ $\forall u \in \mathcal{N}_\lambda$. Therefore, condition (3.5) is always fulfilled in these cases, and it is sufficient for the proof of assertion (1) to prove the existence of a minimizer in problem (3.3).

Let $u_m \in \mathcal{N}_\lambda$ $(m = 1, 2, \ldots)$ be a minimizing sequence in problem (3.3), i.e., let $E_\lambda(u_m) \to \hat{E}_\lambda$ as $m \to \infty$.

First, consider the case $1 < p < \min\{2,q\}$. Using the Hölder and Sobolev inequalities, we obtain

$$E_\lambda(u) \geq \max\left\{\frac{1}{2}T(u) - C\frac{1}{p}T(u)^{p/2}, -C\frac{1}{p}B(u)^{p/q} + \frac{1}{q}B(u)\right\},$$

where $0 < C < +\infty$ is independent of $u \in W(D)$. This implies that $E_\lambda(u)$ is a coercive functional on $W(D)$ and, therefore, there exists a subsequence, which we again denote by (u_m), such that $u_m \rightharpoonup u_\lambda \in W(D)$ weakly in $W(D)$ and, by the Sobolev theorem, $u_m \to u_\lambda$ strongly in $L^\gamma(D)$ for $\gamma \in (1, 2^*)$. Since $p \in (1,2)$, this implies that $A(u_m) \to A(u_\lambda)$ and, therefore,

$$E_\lambda(u_\lambda) \leq \liminf_{m\to\infty} E_\lambda(u_m) = \hat{E}_\lambda, \tag{3.12}$$

$$E'_\lambda(u_\lambda) \leq \liminf_{m\to\infty} E'_\lambda(u_m) = 0. \tag{3.13}$$

Since $E_\lambda(u_m) < 0$ for $m = 1, 2, \ldots$, we have $\hat{E}_\lambda < 0$ and, therefore, $u_\lambda \neq 0$. It is easy to verify that, if one of the inequalities (3.12) or (3.13) holds with equality, then u_λ is a minimizer in problem (3.3). Suppose that $E'_\lambda(u_\lambda) < 0$. Then, there exists an $r > 1$ such that $E'_\lambda(ru_\lambda) = 0$ and $E_\lambda(ru_\lambda) < \hat{E}_\lambda$, which is a contradiction. Thus, we conclude that $E_\lambda(u_\lambda) = \hat{E}_\lambda$ and $E'_\lambda(u_\lambda) = 0$, i.e., u_λ is a minimizer in problem (3.3).

Consider the case $1 < q$ and $\max\{2,q\} < p < 2^*$. In this case, $E_\lambda(u)$ is a coercive functional on $\mathcal{N}_\lambda$. Indeed, if $u \in \mathcal{N}_\lambda$, then $E_\lambda(u) = \frac{p-2}{2p}T(u) + \frac{p-q}{pq}B(u) \to \infty$ as $\|u\|_W \to \infty$. Hence, as before, we conclude that there exists a limiting function $u_\lambda \in W(D)$ satisfying inequalities (3.12) and (3.13). Let us show that $u_\lambda \neq 0$. Assume the converse. Then, $r_m := \|u_m\|_1 \to 0$ and

$$0 = T(v_m) - \lambda r_m^{p-2}A(v_m) + r_m^{q-2}B(v_m) \to 1 \quad \text{as} \quad m \to \infty,$$

where $v_m = u_m/\|u_m\|_1$ for $m = 1, \ldots$. This a contradiction. Let us show that (3.12) and (3.13) hold with equality. Assume the converse. Then $E'_\lambda(u_\lambda) < 0$, and there exists an $r_0 \in (0,1)$ such that $E'_\lambda(r_0u_\lambda) = 0$. Since $r_0 < 1$ and $r = 1$ is a maximizer of the function $E_\lambda(ru_m)$ $(m = 1, \ldots)$, then

$$E_\lambda(r_0u_\lambda) \leq \liminf_{m\to\infty} E_\lambda(r_0u_m) \leq \liminf_{m\to\infty} E_\lambda(u_m) = \hat{E}_\lambda.$$

This and $r_0u_\lambda \in \mathcal{N}_\lambda$ entail a contradiction, which proves assertion (1).

Assertion (2) in the case $1 < q < p \leq 2$ was proved in [11] (also see [19, 20, 23]). Here we prove this assertion in the case $2 < p < q < 2^*$. The nonexistence of solutions to problem (2.1) in the case $0 < \lambda < \lambda_{(p,q)}$ follows from Lemma 3.5. Indeed, any solution belongs to the Nehari manifold, and by Lemma 3.5 we have $\mathcal{N}_\lambda = \emptyset$ if $0 < \lambda < \lambda_{(p,q)}$.

Lemma 3.7. *Let $2 < p < q < 2^*$ and $\lambda > \lambda_{(p,q)}$. Then, there exists a minimizer $u_\lambda \in \mathcal{N}_\lambda$ in problem* (3.3).

Proof. Let $\lambda > \lambda_{(p,q)}$. Then $\mathcal{N}_\lambda$ is not empty by Lemma 3.5. Let $u_m \in \mathcal{N}_\lambda$ $(m = 1, 2, \ldots)$ be a minimizing sequence in problem (3.3), i.e., $E_\lambda(u_m) \to \hat{E}_\lambda$ as $m \to \infty$. We show that u_m is bounded in $\mathcal{D}^{1,2}(\Omega)$. Using the Hölder inequality, we obtain

$$E_\lambda(u_m) \geq -\lambda \frac{1}{p} A(u_m) + \frac{1}{q} c_\Omega (A(u_m))^{q/p}, \quad m = 1, \ldots,$$

where $0 < c_\Omega < +\infty$ is independent of m. This and the inequality $q > p$ imply that $A(u_m)$ is bounded which, in turn, due to $E_\lambda'(u_m) = 0$ implies the boundedness of $T(u_m)$, which was to be proved. In addition, we proved that $\hat{E}_\lambda > -\infty$. Consider the sequence $v_m = u_m/\|u_m\|_1$, $r_m := \|u_m\|_1$ $(m = 1, 2, \ldots)$. Since r_m is bounded, we may assume without loss of generality that $r_m \to r_0$ as $m \to \infty$ for a certain $r_0 \geq 0$. Note that for $m = 1, \ldots$, it holds that

$$\frac{q-2}{2q} T(v_m) - \lambda r_m^{p-2} \frac{q-p}{pq} A(v_m) = \frac{E_\lambda(v_m)}{r_m^2}. \tag{3.14}$$

Taking into account that $-\infty < \hat{E}_\lambda$, we obtain that $r_0 \neq 0$. Since $\|v_m\| = 1$ for $m = 1, 2, \ldots$, we have that, by the Eberlein–Šmulian theorem and the Sobolev embedding theorem, there exists a subsequence, which we again denote by (v_m), such that $v_m \rightharpoonup v_0$ weakly in $\mathcal{D}^{1,2}(\Omega)$ and strongly $v_m \to v_0$ in $L_p(\Omega)$ and $L_q(\Omega)$ for a certain $v_0 \in \mathcal{D}^{1,2}(\Omega)$. Hence we can use (3.14) to prove by contradiction that $v_0 \neq 0$. Then, due to the weak lower semicontinuity of the functional $T(u)$ on $\mathcal{D}^{1,2}(\Omega)$, we obtain

$$E_\lambda(u_\lambda) \leq \liminf_{m\to\infty} E_\lambda(u_m) = \hat{E}_\lambda, \tag{3.15}$$

$$E_\lambda'(u_\lambda) \leq \liminf_{m\to\infty} E_\lambda'(u_m) = 0, \tag{3.16}$$

where $u_\lambda = r_0 v_0$. Assume that there is a strict inequality in (3.16). Then, since $E_\lambda'(u_\lambda) < 0$, there exists a $t > 1$ such that $E_\lambda'(tu_\lambda) = 0$, $E_\lambda''(tu_\lambda) > 0$, and $E_\lambda(tu_\lambda) < E_\lambda(u_\lambda) \leq \hat{E}_\lambda$. This is a contradiction. Therefore, (3.16) holds with equality. Then, $u_\lambda \in \mathcal{N}_\lambda$, which implies the equality in (3.15), which was to be proved.

Now we are ready to complete the proof of assertion (2). Let $\lambda > \lambda_{E,(p,q)}$. Then, since $\lambda_{E,(p,q)} > \lambda_{(p,q)}$, Lemma 3.7 implies the existence of a solution u_λ to the Nehari problem (3.3). Since $\lambda > \lambda_{E,(p,q)}$, by Lemma 3.5 there exists a $u \in \mathcal{D}^{1,2}(D)\backslash 0$ such that $E_\lambda(u) < 0$. This implies that $\hat{E}_\lambda < 0$ and, therefore, $E_\lambda(u_\lambda) < 0$ and $E_\lambda''(u_\lambda) \neq 0$. Hence, u_λ satisfies condition (3.5) and, therefore, Eq. (2.1). This completes the proof of the theorem.

4. LINEAR INSTABILITY

In this section, we discuss some results on the linear instability of stationary solutions to the equation

$$\partial_t u = \Delta u + \lambda |u|^{p-2} u - |u|^{q-2} u, \quad (t, x) \in (0, \infty) \times D. \tag{4.1}$$

Here, as above, $D = \mathbb{R}^N$, $D = \Omega$, or $D = \mathbb{R}^N \backslash \overline{\Omega}$. We consider the solutions subject to boundary conditions (1.2)–(1.4) and the initial condition

$$u|_{t=o} = v_0. \tag{4.2}$$

The solution to problem (4.1), (4.2) will be denoted by $u(t; v_0)$. It is known (e.g., see [28, 29]) that, if $v_0 \in L^\infty(D)$, then there exists a unique classical solution $u(\cdot\,; v_0) \in C^{2,1}([0,T) \times D) \cap C([0,T) \times \overline{D})$ to problem (4.1), (4.2) for a certain $T \equiv T(v_0) \in (0,+\infty)$. Moreover, if $v_0 \in C(\overline{D})$, then $u \in C((0,T),\mathfrak{D}^{1,2}(D))$, and, if additionally $v_0 \in \mathfrak{D}^{1,2}(D)$, then

$$\int_0^t \|u_t(s;v_0)\|_{L^2}^2 ds + E_\lambda(u(t;v_0)) = E_\lambda(v_0). \tag{4.3}$$

In the case $T = +\infty$, the solution $u(\cdot;v_0)$ is said to be *global*.

Let u_0 be a weak bounded in D solution to problem (2.1). Consider the linearized problem

$$\begin{gathered} -L\psi := -\Delta\psi - (\lambda(p-1)|u_0|^{p-2} - (q-2)|u_0|^{q-1})\psi = \mu\psi \\ \psi \in W(D). \end{gathered} \tag{4.4}$$

Then, in the case $2 < \min\{p,q\} < +\infty$, it is easy to verify that there exists the minimal eigenvalue μ_1 in problem (4.4) with the nonnegative eigenfunction $\psi_1 \in W(D)$ (e.g., see [26, 30]).

The solution u_0 to problem (1.1) is called a *linearly unstable stationary solution* to problem (4.1) if the minimal eigenvalue μ_1 of the operator $-L$ is negative. Note that, if we consider the perturbed solution $v(t,x) = u_0(x) + w(t,x)$ to problem (4.1), then upon the linearization we obtain $\partial_t w = Lw$. Hence, for $w = e^{-t\mu_1}\psi_1$, we see that the perturbation is exponentially unstable in the linear approximation.

Lemma 4.1. *Let $N \ge 1$, $p \ne q$, $p \ge 2$, $q \ge 2$, and Ω be a bounded star-shaped (with respect to the origin of $\mathbb{R}^N$) domain with a C^1-smooth boundary $\partial\Omega$. Then, the weak bounded in D solution u to problem* (2.1) *is a linearly unstable stationary solution to problem* (4.1) *if the following conditions are satisfied:*

(i) $D = \mathbb{R}^N$ *and* $d^*(p,q) < 0$;

(ii) $D = \Omega$ *and* $\max\{2,q\} < p < 2^*$;

(iii) $D = \mathbb{R}^N \setminus \Omega$ *for* $2 < p < q$ *or* $d^*(p,q) < 0$ *and* $0 < q < p$.

Proof. Note that, under conditions (i)–(iii), the weak solution u to problem (2.1) by Theorems 2.2, 2.4, and 2.5 satisfies the inequality $E''(u) < 0$. By the minimax Courant–Fischer principle, we have

$$\mu_1 = \inf_{\psi \in D^{1,2}(D)\setminus\{0\}} \frac{\int_D (|\Delta\psi|^2 - (\lambda(p-1)|u|^{p-2} - (q-2)|u|^{q-1})\psi^2)dx}{\int_D |\psi|^2 dx}. \tag{4.5}$$

Set $\psi = u$. Then, we have

$$\frac{\int_D \left(|\nabla u|^2 - (\lambda(p-1)|u|^{p-2} - (q-1)|u|^{q-2})u^2\right)dx}{\int_D |u|^2 dx} = \frac{E''(u)}{\int_D |u|^2 dx}. \tag{4.6}$$

Hence, due to the inequality $E''(u_\lambda) < 0$, (4.5) implies $\mu_1 < 0$, which proves the lemma.

Note that the approach based on the use of the fibering function $E(ru)$ allows us to obtain the following result on the upper bound on the minimal eigenvalue of the linearized problem (4.4).

Corollary 4.2. *Let $N \ge 1$, $p \ne q$, $p \ge 2$, $q \ge 2$, $D = \mathbb{R}^N$, $D = \Omega$, or $D = \mathbb{R}^N \setminus \overline{\Omega}$, where Ω is a bounded domain with a C^1-smooth boundary $\partial\Omega$. Let u be a weak bounded in D solution to problem* (2.1). *Then, the minimal eigenvalue μ_1 of the linearized problem* (4.4) *satisfies the inequality*

$$\mu_1 \le \frac{E''(u)}{\int_D |u|^2 dx}.$$

Proof. The proof immediately follows from the minimax Courant–Fischer principle (4.5) and formula (4.6).

5. ON THE STABILITY OF GROUND STATES

Note that Derrick's theorem (see [15]) contains a stronger assertion then in Lemma 4.1 under condition (i); namely, for $D = \mathbb{R}^N$, $N \geq 3$ and $p > 2$, $q > 2$, if there exists a weak bounded in $\mathbb{R}^N$ solution u to problem (2.1), then it is a linearly unstable stationary solution to problem (4.1). It was shown in [16] that the same result holds in the case $N = 1, 2$ if $u \in H^\infty(\mathbb{R}^N)$. However, the situation can be different if $p, q \in (1, 2]$. In particular, this can be due to the fact that problem (2.1) in this case can have solutions with compact support (see [6, 19, 20, 31]). Below, we show that such solutions are, in a certain sense, stable in the case $D = \mathbb{R}^N$. The situation becomes different in the case of a bounded domain $D = \Omega$. In this case, problem (2.1) has stable solutions. In the case $p, q \in (1, 2]$, such a result was obtained in [11]. In this section, we extend this result to other exponents.

In what follows, the solution u_0 to problem (2.1) is called a $\mathcal{D}^{1,2}(D)$-*stable* stationary solution to the parabolic problem (4.1), (4.2) if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\|u_0 - u(t; v_0)\|_1 < \varepsilon \quad \forall v_0 \in \mathcal{D}^{1,2}(D) \cap C(\overline{D}): \ \|u_0 - v_0\|_1 < \delta, \quad \forall t > 0. \tag{5.1}$$

At a fixed $\lambda > 0$, the set of all ground states of problem (2.1) will be denoted by $g_\lambda := g_\lambda(D)$, and it will be called the *manifold of ground states*.

Proposition 5.1. *Let $2 < p < q < 2^*$, $\lambda > \lambda_{(p,q)}$, and $D = \Omega$ be a bounded domain in $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, $g_\lambda(D)$ is a bounded set in $\mathcal{D}^{1,2}(D)$.*

Proof. This proposition is proved by contradiction and the proof is similar to the proof of Lemma 3.7.

Let $\delta > 0$. Introduce the notation $V_\delta(g_\lambda) := \{v \in \mathcal{D}^{1,2}(D): \ \inf_{u \in g_\lambda} \|u - v\|_1 < \delta\} \cap C(\overline{D})$. The manifold of ground states g_λ will be called $\mathcal{D}^{1,2}(D)$-*stable* for the parabolic problem (4.1), (4.2) if, $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that

$$\inf_{u_0 \in G_\lambda} \|u_0 - u(t; v_0)\|_1 < \varepsilon \quad \forall v_0 \in V_\delta(G_\lambda), \ \ \forall t > 0. \tag{5.2}$$

Lemma 5.2. *Let $N \geq 1$, $1 < q < p < 2$ or $2 < p < q < 2^*$, and $D = \Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, if $\lambda > \lambda_{(p,q)}$, then the manifold of ground states g_λ of problem* (2.1) *is $\mathcal{D}^{1,2}(D)$-stable.*

Proof. Consider the manifold of ground states g_λ of problem (2.1). Note that by Theorem 3.6, if $\lambda > \lambda_{(p,q)}$, then $E''_\lambda(u) > 0$ for every $u \in g_\lambda$. Since g_λ is a bounded set in $\mathcal{D}^{1,2}(D)$ and the mappings $E_\lambda, E''_\lambda: \mathcal{D}^{1,2}(D) \to \mathbb{R}$ are continuous, there exists a $\delta_0 > 0$ such that, for all $u \in V_\delta(g_\lambda)$ and $0 < \delta < \delta_0$, it holds that $E''_\lambda(u) > 0$.

Let us show that E_λ is a Lyapunov functional in the neighborhood $V_\delta(g_\lambda)$ for $0 < \delta < \delta_0$.

Proposition 5.3. *There exists a $\delta \in (0, \delta_0)$ such that*

$$E_\lambda(u) > \hat{E}_\lambda \quad \forall u \in V_\delta(g_\lambda) \backslash g_\lambda. \tag{5.3}$$

Proof. Assume the converse, i.e., let, for each $\delta \in (0, \delta_0)$, there exists a $u^\delta \in V_\delta(g_\lambda) \backslash g_\lambda$ such that $E_\lambda(u^\delta) \leq \hat{E}_\lambda$. Then, there exists a sequence $u_n \in V_{\delta_0}(g_\lambda) \backslash g_\lambda$ such that

$$\inf_{u \in g_\lambda} \|u - u_n\|_1 \to 0 \quad \text{as} \quad n \to \infty$$

and

$$E_\lambda(u_n) \leq \hat{E}_\lambda, \quad n = 1, 2, \dots . \tag{5.4}$$

The first convergence implies that there exists a sequence $v_n \in g_\lambda$ such that

$$\|v_n - u_n\|_1 \to 0 \quad \text{as} \quad n \to \infty. \tag{5.5}$$

Note that (v_n) is a minimizing sequence for (3.3) because $E_\lambda(v_n) \equiv \hat{E}_\lambda$ for all $n = 1, 2, \ldots$. Therefore, we can use the same reasoning that was used for the minimizing sequence in the proof of Lemma 3.7. Therefore, (v_n) has a limiting point $u_a \neq 0$, which is a ground state of problem (2.1). Furthermore, $v_m \to u_\lambda$ strongly in $\mathfrak{D}^{1,2}(\Omega)$ as $n \to \infty$. This and (5.5) imply that $u_n \to u_\lambda$ strongly in $\mathfrak{D}^{1,2}(\Omega)$ as $n \to \infty$. Note that by construction u_n is not a ground state of (2.1). Therefore,

$$E_\lambda(r_{\min}(u_n)u_n) > \hat{E}_\lambda, \quad n = 1, 2, \ldots .$$

This inequality and (5.4) imply

$$1 < r_{\max}(u_n) < r_{\min}(u_n). \tag{5.6}$$

Since the mappings $r_{\max}(\cdot), r_{\min}(\cdot) : \mathfrak{D}^{1,2}(\Omega) \to \mathbb{R}$ are continuous and $u_n \to u_\lambda$ in $\mathfrak{D}^{1,2}(\Omega)$ as $n \to \infty$, we have

$$r_{\min}(u_n) \to r_{\min}(u_\lambda) = 1 \quad \text{as} \quad n \to \infty.$$

Here $r_{\min}(u_\lambda) = 1$ because $E_\lambda''(u_\lambda) > 0$. Then, (5.6) implies that

$$r_{\max}(u_n) \to r_{\min}(u_\lambda) = 1 \quad \text{as} \quad n \to \infty.$$

This and the inequalities $E_\lambda''(r_{\max}(u_n)u_n) \le 0$ and $E_\lambda''(r_{\min}(u_n)u_n) \ge 0$ entail

$$E_\lambda''(u_\lambda) = 0.$$

However, this contradicts the inequality $E_\lambda''(u) > 0$ for all $u \in g_\lambda$.

To complete the proof of Lemma 5.2, it suffices to prove the following proposition.

Proposition 5.4. *For each* $\varepsilon > 0$, *there exists a* $\delta \in (0, \delta_0)$ *such that*

$$\inf_{u \in g_\lambda} \|u - u(t; v_0)\|_1 < \varepsilon \quad \forall v_0 \in V_\delta(g_\lambda), \quad \forall t > 0. \tag{5.7}$$

Proof. Let $\varepsilon \in (0, \delta_0)$. Consider

$$d_\varepsilon := \inf\{E_\lambda(w) : w \in \mathfrak{D}^{1,2}(\Omega), \inf_{u \in g_\lambda} \|u - w\|_1 = \varepsilon\}. \tag{5.8}$$

Then $d_\varepsilon > \hat{E}_\lambda$. Indeed, assume the converse, i.e., let there exist a sequence $w^n \in \mathfrak{D}^{1,2}(\Omega)$ such that $\inf_{u \in g_\lambda} \|u - w^n\|_1 = \varepsilon$ and $E_\lambda(w^n) \to \hat{E}_\lambda$. Since the set g_λ is bounded in $\mathfrak{D}^{1,2}(\Omega)$, (w^n) is also bounded in $\mathfrak{D}^{1,2}(\Omega)$. Then, by the Eberlein–Šmulian theorem and Sobolev's embedding theorem, there exists a $v_0 \in \mathfrak{D}^{1,2}(\Omega)$ and a subsequence (which is again denoted by (w^n)) such that $w^n \rightharpoonup v_0$ weakly in $\mathfrak{D}^{1,2}(\Omega)$ and strongly in L_γ, $1 < \gamma < 2^*$. The weak lower semicontinuity of the functional $\|u\|_1^2$ on $\mathfrak{D}^{1,2}(\Omega)$ implies $\hat{E}_\lambda \ge E_\lambda(v_0)$ and $\inf_{u \in g_\lambda} \|u - v_0\|_1 \le \varepsilon$. By Proposition 5.3, this is possible only if $v_0 \in g_\lambda$. Then, the equality $\hat{E}_\lambda = E_\lambda(v_0)$ implies the strong convergence $w^n \to v_0$ in $\mathfrak{D}^{1,2}(\Omega)$. Hence, $\varepsilon = \inf_{u \in g_\lambda} \|u - w^n\|_1 \to \inf_{u \in g_\lambda} \|u - v_0\|_1$. Then, $v_0 \notin g_\lambda$, which is a contradiction.

Since $\hat{E}_\lambda < d_\varepsilon$, we have $\hat{E}_\lambda < d_\varepsilon - \sigma$ for a certain $\sigma > 0$. The continuity of the mapping $E_\lambda : \mathfrak{D}^{1,2}(\Omega) \to \mathbb{R}$ implies that there exists a $\delta \in (0, \varepsilon)$ such that

$$E_\lambda(w) < d_\varepsilon - \sigma \quad \forall w \in V_\delta(g_\lambda) \subset V_\varepsilon(g_\lambda). \tag{5.9}$$

To prove the proposition, it remains to verify that, for every $v_0 \in V_\delta(g_\lambda)$, the solution $u(t, v_0)$ remains in $V_\varepsilon(g_\lambda)$ for all $t > 0$. Assume the converse. Then, taking into account that $u(t, v_0) \in C((0, T), \mathfrak{D}^{1,2}(\Omega))$, there exists a $t_0 > 0$ such that $\inf_{u \in g_\lambda} \|u - u(t_0, v_0)\|_1 = \varepsilon$. Due to (5.8), this implies

$$d_\varepsilon \le E_\lambda(v(t_0, v_0)).$$

On the other hand, (4.3) implies that $E_\lambda(v(t_0,v_0)) \le E_\lambda(v_0)$. Therefore, taking into account (5.9), we have

$$d_\varepsilon \le E_\lambda(v(t_0,v_0)) \le E_\lambda(v_0) < d_\varepsilon - \sigma.$$

Thus, we have a contradiction, which proves Proposition 5.4.

Let $D = \Omega$. If a nonnegative solution u to problem (2.1) satisfies the condition

$$\partial u/\partial \nu = 0 \quad \text{on} \quad \partial\Omega, \tag{5.10}$$

then u is called the *solution with a compact support.* Note that, according to the standard regularity theory of solutions to elliptic problems, if $u_\lambda \in W(D)$ is a weak solution to (2.1), then $u \in C^2(D) \cap C^{1,\kappa}(\overline{D})$ for $\kappa \in (0,1)$.

In [32, 33], the following lemma was proved.

Lemma 5.5. *Let $N \ge 3$, $1 < q < p < 2$, $d^*(p,q) > 0$, and $D = \Omega$ be a bounded star-shaped domain with respect to the origin of $\mathbb{R}^N$ with a C^1-smooth boundary $\partial\Omega$. Then, there exists a $\lambda^* \equiv \lambda^*_{(p,q)} > \lambda_{(p,q)}$ such that, for all $\lambda \ge \lambda^*$, problem* (2.1) *has a solution u^c_λ with a compact support. Moreover, for $\lambda = \lambda^*_{(p,q)}$, the solution $u^c_{\lambda^*}$ with a compact support is a ground state of problem* (2.1). *In this case, $u^c_\lambda \in C^2(D) \cap C^{1,\kappa}(\overline{D})$ for $\kappa \in (0,1)$ and $u^c_\lambda \ge 0$ in D.*

Note that $\{(p,q) \in \mathbb{R}^2 : 1 < q < p < 2\} \cap \{(p,q) \in \mathbb{R}^2 : d^*(p,q) > 0\} = \emptyset$ for $N = 1, 2$.

Below, we will need the following lemma (see [34]).

Lemma 5.6 (Serrin–Zou). *Let $N \ge 2$, $1 < q < p < 2$, and $D = \mathbb{R}^N$. Then, any solution u to problem* (2.1) *has a compact support. Moreover, for each connected component Ξ in the open support $\Theta := \{x \in \mathbb{R}^N : u(x) > 0\}$, it holds that*

(1) *Ξ is a ball;*

(2) *u is a radially symmetric function with respect to the center of the ball Ξ.*

Let $N \ge 3$ and $1 < q < p < 2$. Consider problem (2.1) for $D = \mathbb{R}^N$ and $\lambda = 1$. Then, by Lemma 3.1, there exists a classical nonnegative solution u^c of this problem that is a ground state of (2.1). Since u^c is a ground state, it is easy to conclude from Lemma 5.6 that the support $\mathrm{supp}(u^c)$ consists of a single component, which is a ball centered (without loss of generality) at zero with a certain radius $R_{(p,q)} > 0$; i.e., $\mathrm{supp}(u^c) = B_{R_{(p,q)}}$. The function u^c is radially symmetric. Note that u^c is a classical solution to problem (2.1) in the case $D = B_{R_{(p,q)}}$. As was shown in [20, 21], this implies that u^c is a unique positive solution to this problem, and the radius $R_{(p,q)}$ is determined uniquely. On the other hand, by Lemma 5.5, if additionally $d^*(p,q) > 0$, then there exists a $\lambda^*_{(p,q)}(B_{R_{(p,q)}})$ such that (2.1) has a ground state $u^c_{\lambda^*}$ with a compact support. Using the uniqueness of u^c and the fact that u^c is a ground state of problem (2.1) for $D = \mathbb{R}^N$, it is easy to verify that $\lambda^*_{(p,q)}(B_{R_{(p,q)}}) = 1$ and $u^c = u^c_{\lambda^*}$. This, in particular, implies the following result.

Corollary 5.7. Let $N \ge 3$, $D = B_{R_{(p,q)}}$, $1 < q < p < 2$, and $d^*(p,q) > 0$. Then, the manifold of ground states $g_{\lambda^*}(B_{R_{(p,q)}})$ of problem (2.1) with $\lambda = 1$ consists of a single solution u^c, and u^c has a compact support.

Every function w in $\mathcal{D}^{1,2}(\Omega)$ can be extended to $\mathbb{R}^N$ by the formula

$$\begin{aligned} \tilde{w} &= w \quad \text{in} \quad \Omega, \\ \tilde{w} &= 0 \quad \text{in} \quad R^N \backslash \Omega. \end{aligned} \tag{5.11}$$

Then, $\tilde{w} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, and in this sense $\mathcal{D}^{1,2}(\Omega) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Lemma 5.2 and Corollary 5.7 imply the following result.

Lemma 5.8. *Let* $N \geq 3$, $D = \mathbb{R}^N$, $1 < q < p < 2$, *and* $d^*(p,q) > 0$. *Then, the compactly supported solution* u^c *to problem* (1.1)–(1.3) *is a stable stationary state of the parabolic problem* (4.1)–(4.2) *in the following sense:* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that*

$$\|u^c - u(t;v_0)\|_1 < \varepsilon \quad \forall v_0 \in \mathcal{D}^{1,2}(B_{R_{(p,q)}}) \cap C(\overline{D}) : \|u^c - v_0\|_1 < \delta, \quad \forall t > 0. \tag{5.12}$$

It is clear from the translational invariance of Eqs. (1.1), (4.1) that $u^c(\cdot + y)$ also is a stable stationary state of the parabolic problem (4.1), (4.2) for every $y \in \mathbb{R}^N$ in the sense similar to Lemma 5.8. In particular, consider

$$\mathcal{M}_\delta := \{v \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \exists y \in \mathbb{R}^N,\ v(\cdot + y) \in \mathcal{D}^{1,2}(B_{R_{(p,q)}}),\ \sup_{y \in \mathcal{O}_v} \|u^c(\cdot) - v(\cdot + y)\|_1 < \delta\},$$

where $\delta > 0$ and $\mathcal{O}_v := \{y \in \mathbb{R}^N : v(\cdot + y) \in \mathcal{D}^{1,2}(B_{R_{(p,q)}})\}$ for $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, we obtain the following result.

Corollary 5.9. Let $N \geq 3$, $D = \mathbb{R}^N$, $1 < q < p < 2$, and $d^*(p,q) > 0$. Then, there exists a $\delta_0 > 0$ such that, for all $v_0 \in \mathcal{M}_{\delta_0}$, the solution $u(t;v_0)$ to problem (4.1), (4.2) is global and bounded for all $t > 0$.

6. CONCLUDING REMARKS

It is easy to see that Theorem 3.6 on the existence of solutions in the case of a bounded domain $D = \Omega$ can be extended to Eqs. (1.1) with a nonzero mass, at least, for small $m > 0$. Taking into account Theorem 2.4, the existence of solutions to Eq. (1.1) in the case $D = \Omega$ when $2^* \leq p < q$, $0 < p < q \leq 1$, or $0 < q < p \leq 1$ remains an open question.

I am not aware of the results on the existence of solutions to Eq. (1.1) in the case $D = \mathbb{R}^N \setminus \Omega$ in the same generality as in the case of the two other domains if two exponents p and q are taken into account. Some results in this direction can be found in [35–37] (also see the references therein).

I believe that the results obtained in Sections 4 and 5 can be extended to other types of nonstationary equations—hyperbolic, nonlinear Schrödinger equations, etc.

Note that Derrick in [15] concluded from the result on the instability of localized solutions that this is an obstacle for interpreting such solutions as particles. In [15], he proposed a few modifications of models aimed at obtaining stable localized solutions. Lemma 5.8 and Corollary 5.9 suggest that equations with non-Lipschitz nonlinearities can be considered as a way to obtain models with stable localized solutions.

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